# A genetic algorithm-based method for synthesis of low peak amplitude signals

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The maximum amplitude of a waveform corresponding to a particular harmonic spectrum depends on the phases of its harmonic components. A waveform with a low peak-to-rms ratio is desirable in situations requiring a maximum signal-to-noise ratio. This paper introduces a genetic algorithm-based method for selecting the phases that produces better results than previously described methods. Results for four different amplitude spectra are given. For the case of a flat spectrum with up to 40 harmonics, the genetic algorithm finds peak factors (peak/$\sqrt{2}$ rms) ranging from 0.98 to 1.24. © *1996 Acoustical Society of America.*

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## INTRODUCTION

Many applications of computer music, signal processing, and communications benefit from minimizing the peak amplitude of a periodic signal. For instance, wavetable synthesis stores one period of a waveform in memory and uses table lookup to find the sample values. If the average or rms amplitude of the waveform is small compared to its peak amplitude, a poor signal-to-quantization noise ratio results. Conversely, a waveform with phases carefully chosen to achieve a minimum peak amplitude will have a maximal signal-to-noise ratio.

A closed-form solution of the peak amplitude problem does not exist. However, several researchers have investigated *ad hoc* solutions to the problem (Schroeder, 1970; Pumplin, 1985; van den Bos, 1987). Schroeder and van den Bos report their results with respect to a peak-to-peak factor rather than the maximum absolute peak. With the discrete waveform defined by

$$s_n=\sum_{k=1}^{K} a_k \cos(2\pi k f_1 n/N+\phi_k), \tag{1}$$

where $N \gg K$, the peak-to-peak factor is defined as

$$\text{peak-to-peak factor}=\frac{\max(s_n)+\max(-s_n)}{2\sqrt{2}s_{\text{rms}}}, \tag{2}$$

where

$$s_{\text{rms}}=\left(0.5\sum_{k=1}^{K} a_k^2\right)^{1/2}. \tag{3}$$

However, for reasons given below, we prefer the following definition:

$$\text{peak factor}=\max|s_n|/\sqrt{2}s_{\text{rms}}, \tag{4}$$

which is equal to the peak amplitude $(\max(|s_n|))$ when the harmonic amplitudes are normalized so that the sum of their squares equals 1.0.

When the minimum and maximum signal values happen to have the same absolute value (e.g., for waveforms having odd symmetry), the two definitions give identical results. Note that for a sine wave ($K=1$), both the peak factor and peak-to-peak factor are 1.0. More often, however, the criteria are different. The peak factor criterion is more stringent in the sense that it looks at only the worst peak rather than averaging the positive and negative peaks. A difference between the criteria occurs in applications like digital wavetable synthesis where the signal-to-noise ratio directly depends on the maximum absolute peak amplitude. Introducing a dc offset corresponding to the average of the minimum and maximum peak values in order to equalize these peaks can cause problems: audible "clicks" when the signal begins and ends. Most of the numerical examples presented later will include both peak factor (peak amplitude) and peak-to-peak factor measures for comparison.

## I. EXISTING METHODS

Various strategies exist for choosing the harmonic phases to produce low peak amplitudes. Setting all the phases to zero gives the worst case and largest peak factor. With zero phases, all peaks of the individual cosines line up, so the maximum signal value equals the sum of the partial amplitudes ($\sqrt{K}$ for flat spectra). Exhaustive search and use of minimax search strategies (van den Bos, 1987) are only practical for spectra having small numbers of harmonics.

Picking random phases (Pumplin, 1985; van den Bos, 1987) provides a reasonable starting point for finding reduced peak factors. Extending this idea, a program could generate several sets of random phases and then select the set with the lowest corresponding peak amplitude.

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To improve on random phase selection, Schroeder (1970) derived a simple and intuitive formula for the phases as follows:

$$\phi_k = \phi_1 - 2\pi \sum_{j=1}^{k-1} (k-j) a_j^2, \quad \text{for } k = 2,3,\ldots,K, \tag{5}$$

where $\phi_1$ can be an arbitrary value and $a_j$ is the normalized amplitude of the $j$th harmonic partial. Schroeder's formula makes some assumptions about the signal's spectrum. If these assumptions are not met, it may not perform well. These assumptions include a small spectral bandwidth relative to the center frequency, and a smooth amplitude spectrum. Spectra with sharp resonances and long exponential tails of upper harmonics fail to meet these assumptions.

A more recent method proposed by van den Bos (1987) does not make assumptions about the spectrum of the signal. Van den Bos's technique attempts to select the phase angles so that the signal is maximally similar to a two-level signal. The technique starts with random initial phases and uses a discrete Fourier transform-based iterative procedure to converge to a local minimum. The procedure significantly improves on Schroeder's approximation at the modest cost of the iterative computation. However, the procedure converges to a local optimum that depends on the initial conditions. Van den Bos recommends repetition of the procedure several different times with different sets of initial phases.

Similarly, Pumplin (1985) proposed a gradient search of the phase space where minimization of $(1/N)\Sigma_{n=0}^{N-1} s_n^4$, the signal's fourth moment, is the criterion. As with van den Bos's technique, simple gradient search converges to the nearest local optimum, and like van den Bos, Pumplin suggests repetition of the procedure with different initial phases to achieve better results.

None of the above procedures guarantees an optimal solution. Can another procedure find better results? The next section addresses this question.

## II. A GENETIC ALGORITHM-BASED PROCEDURE

The method for peak amplitude minimization proposed in this section uses a genetic algorithm (Goldberg, 1989; Holland, 1975) to select the phase angles. Genetic algorithms (GAs) employ natural selection and evolutionary-inspired operators like crossover and mutation to optimize combinatorial problems such as the phase angle alignment task.

GAs work with a population of candidate solutions to efficiently examine the search space and explore its various local optima in parallel. This robust parallel exploration is well suited to the phase alignment problem where it is better to avoid converging to local minima. We can use the GA to systematically explore the search space rather than start the procedure over with several different initial conditions.

Since GA methods avoid simple convergence to local optima, we can seed the initial population with approximations to phase values found by van den Bos's procedure and then proceed. The advantage is faster convergence, though there is a risk that the GA may neglect to explore promising portions of the search space if the population converges too quickly to the seeded solutions. An expedient solution is to perform a quick run with seeding and resort to a second unseeded run only if the seeded GA does not find a solution significantly better than the seeds.

To use a GA, we must specify an objective function and a method for coding the phases. The discrete-time signal that we wish to have a low peak amplitude is given by Eq. (1), which we repeat:

$$s_n = \sum_{k=1}^{K} a_k \cos(2\pi k f_1 n/N + \phi_k), \tag{6}$$

where $a_k$ and $\phi_k$ are the amplitude and phase of the $k$th harmonic, respectively, $N$ is the number of samples per period and $K$ is the number of harmonics. The problem is then to minimize

$$\Psi(\{\phi_k\}) = \max |s_n|, \quad 0 \leq n < N \tag{7}$$

by choosing $\{\phi_k\}$ such that $0 \leq \phi_k < 2\pi$ for $k = 2,3,\ldots,N$ (without loss of generality, we set $\phi_1$ to zero).

Now let us consider the encoding of the phases. Most GAs usually work with binary-valued encoded parameters. We can meet this requirement by introducing an integer-valued parameter $p_k$ that indexes sampled values of the phases. The discrete phase version of Eq. (6) is

$$s_n = \sum_{k=1}^{K} a_k \cos(2\pi k f_1 n/N + 2\pi p_k/L), \tag{8}$$

and we now wish to minimize

$$\Psi(\{p_k\}) = \max |s_n|, \quad 0 \leq n < N \tag{9}$$

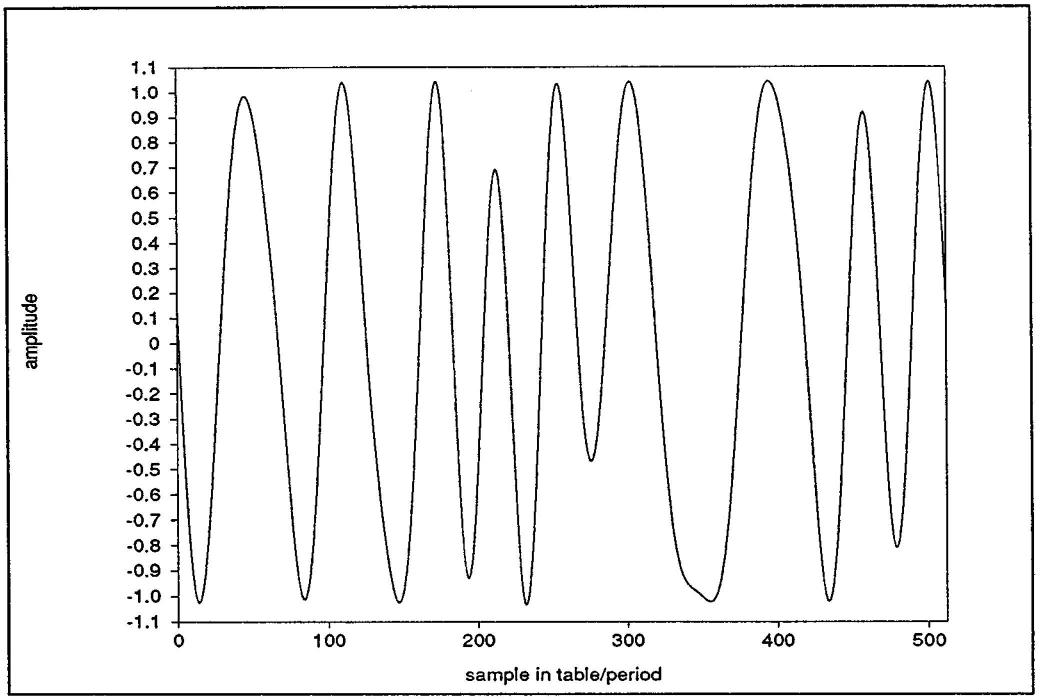
by choosing $\{p_k\}$ such that $0 \leq p_k < L = 2^M$. With $L$ taken as a power of two, it is natural to interpret $p_k$ as a binary integer for encoding. Equations (8) and (9) define the objective function that the GA must minimize.

Our procedure uses a standard simple genetic algorithm with binary tournament selection, one-point crossover, and a mutation rate equal to the reciprocal of the number of bits in the encoding. Goldberg (1989) gives more details on the workings of genetic algorithms. Numerical results for the GA-based technique and the other methods are presented and compared in the next section.

The genetic algorithm is a relatively efficient iterative procedure. The speed of the GA directly depends on the number of phases. For a few phases, the GA completes in just a couple seconds on a Silicon Graphics Indigo workstation (the Indigo benchmarks at approx. 100 000 Dhrystones/second); for 50 or more harmonics, the GA requires several

TABLE I. Peak amplitudes and peak-to-peak factors for example 1.

| | Peak amplitude | Peak-to-peak factor |
|---|---|---|
| worst | 3.61 | 3.01 |
| $avg_{100}$ random | 1.88 | 1.72 |
| $min_{100}$ random | 1.39 | 1.39 |
| Schroeder | 1.24 | 1.17 |
| van den Bos | 1.10 | 1.07 |
| GA | 1.04 | 1.04 |




$\{\phi_k\} = \{0.000000, 0.000000, 3.151593, 4.994641, 0.961748, 2.840932, 3.446117, 1.061515, 0.517010, 0.088175, 2.350059, 0.681087, 2.860932, 1.466486, 4.773748, 3.111593\}$

FIG. 1. Waveform and phase values for example 1 with GA-minimized peak amplitude.

minutes. Though it is slower than van den Bos's iterative procedure, it usually does not need to be repeated with different initial conditions.

## III. NUMERICAL EXAMPLES AND RESULTS

This section presents four numerical examples and gives the peak amplitudes and peak-to-peak factors for the Schroeder, van den Bos, and GA methods. The first two examples were originally proposed by Schroeder (1970), while the third was given by van den Bos (1987). These earlier examples use simple theoretical spectra to compare how the methods performed. We expand on the second example (a flat spectrum) by showing how the peak amplitude varies with the number of harmonics in the spectrum.

We also give a fourth example based on a measured tenor voice spectrum. If we were trying to model a tenor voice by wavetable synthesis, we would likely load one or more wavetables with measured tenor spectra. Thus this example represents a typical problem one might encounter in sound synthesis.

In each case the spectrum is normalized so that the sum of the squared harmonic amplitudes equals 1. With normalization, the peak amplitude corresponds to the peak factor as defined in Eq. (4). Since the rms amplitude is always the same for normalized spectra (0.7071), this allows comparison of the peak amplitudes from different signals.

Following van den Bos's lead, we use $N=512$ samples per period. This allows more than ten samples per period for the highest harmonic used in the examples ($K=50$).

**Example 1:** Schroeder gives the following normalized amplitude spectra as his first example (1970):

$$a_k=(1/\sqrt{8})\ \sin[(2k-1)/32], \quad k=1,2,\ldots,16. \tag{10}$$

Table I summarizes the peak amplitudes found by the various methods. The worst case occurs when all the phases are set to zero, giving a large peak of 3.61. Generating 100 sets of random phases produced an average peak amplitude of 1.88 and a minimum of 1.39, substantial improvements over the worst case. Schroeder's phase formula yields further improvement (1.24), and van den Bos's technique does even better by returning 1.10. The GA-based method, using a resolution of $L=64$ values (an encoding of $M=6$ bits) per phase, achieves the lowest peak amplitude (1.04). Figure 1 shows the resulting signal and gives the actual phases used for the GA-derived case.

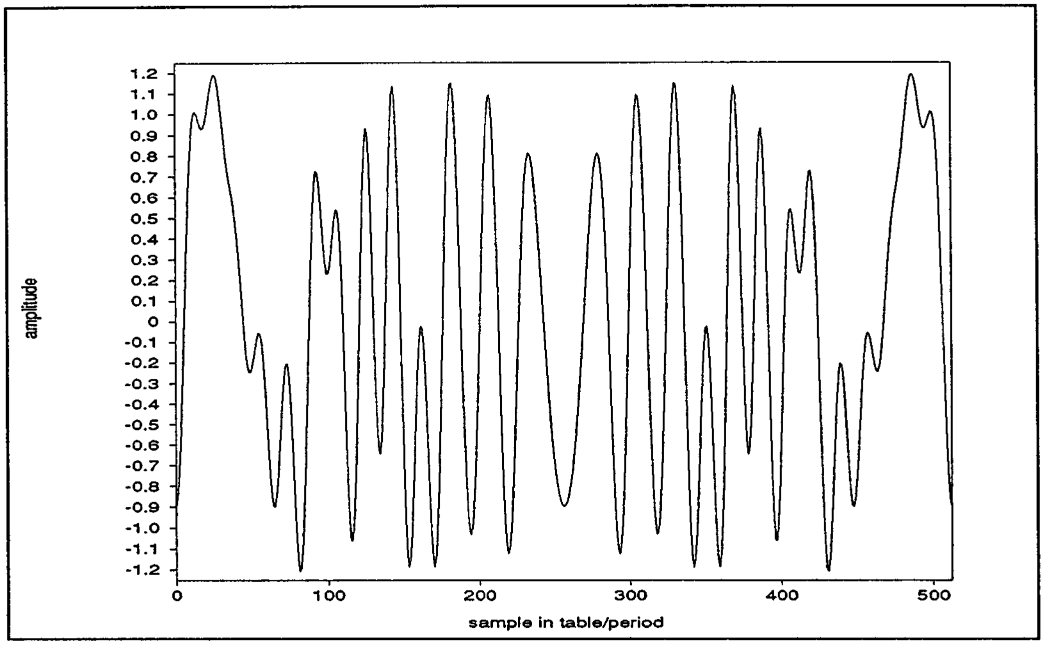
TABLE II. Peak amplitudes and peak-to-peak factors for example 2 with phases restricted to values of 0 or $\pi$.

| | Peak amplitude | Peak-to-peak factor |
|---|---|---|
| worst | 5.57 | 3.44 |
| $\text{avg}_{100}$ random | 1.90 | 1.73 |
| $\text{min}_{100}$ random | 1.54 | 1.42 |
| Schroeder | 1.41 | 1.38 |
| van den Bos | 1.26 | 1.26 |
| GA | 1.21 | 1.20 |

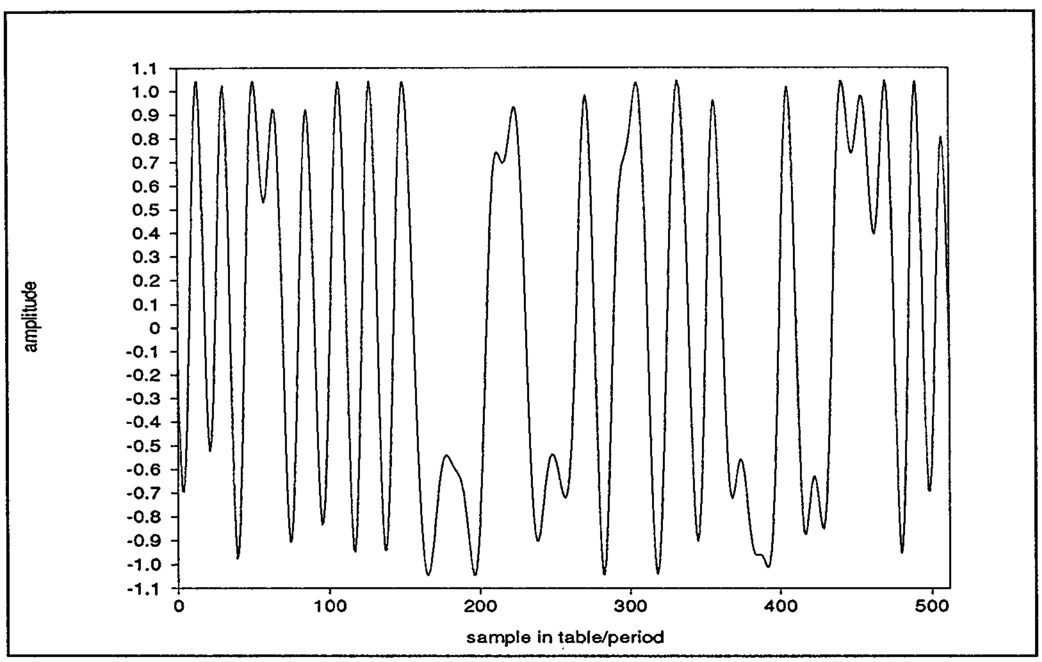
Table I also shows the peak-to-peak factors for each of the methods. For example 1, the two criteria give similar results. The solutions found by the GA differ under the two criteria, but only slightly.

**Example 2:** Schroeder gives a flat spectrum for his second example:

$$a_k = 1/\sqrt{K}, \quad k = 1,2,\ldots,K, \tag{11}$$




$\{\phi_k\} = \{0, 0, 0, 0, 0, \pi, \pi, \pi, \pi, \pi, 0, \pi, \pi, \pi, 0, \pi, 0, \pi, \pi, 0, \pi, 0, \pi, \pi, 0, \pi, \pi, 0, 0, \pi, \pi\}$




$\{\phi_k\} = \{0.000000, 5.688001, 1.871457, 2.681398, 4.540583, 3.123185, 0.196350, 4.301282, 0.312932, 6.258642, 2.258020, 4.135612, 3.092505,$
$5.454836, 1.846913, 2.577088, 0.184078, 3.184544, 5.295302, 0.104311, 4.154020, 3.485204, 0.546097, 0.951068, 4.945554, 2.675262,$
$2.264156, 1.595340, 1.435806, 1.484893, 2.135301\}$

FIG. 2. (a) Waveform and 0 or $\pi$ phases for example 2 with GA-minimized peak amplitude. (b) Waveform and unrestricted phase values for example 2 with GA-minimized peak amplitude. (c) Minimum peak amplitudes found by the various techniques for a flat spectra with different numbers of harmonics. (d) Waveform of 14-harmonic flat spectrum found by GA with a peak amplitude of 0.9795.

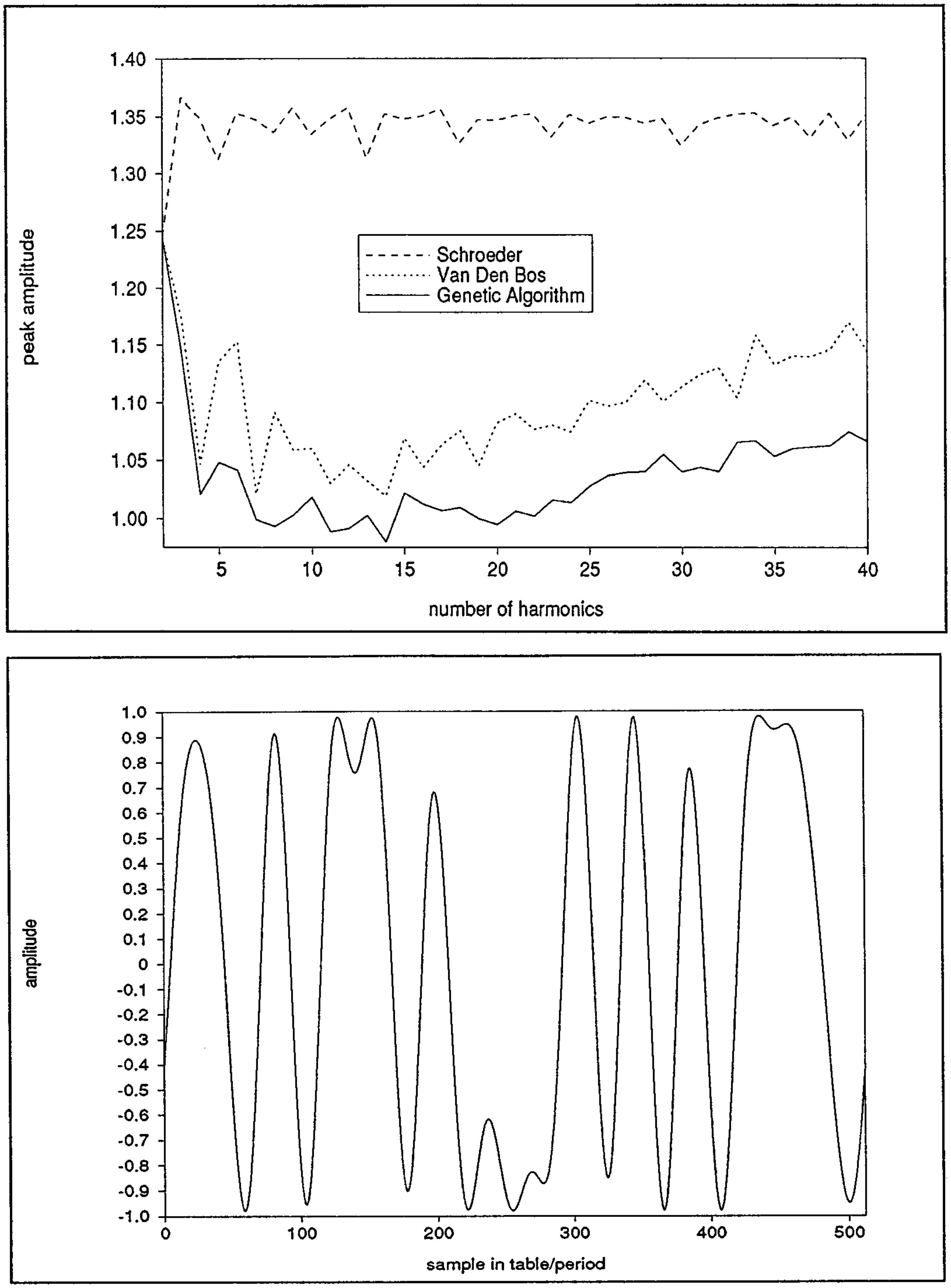



FIG. 2. (*Continued.*)

with $K = 31$.

Schroeder also limits the phases to the discrete values 0 and $\pi$ in order to produce a waveform with even symmetry. This condition is easily met with the GA-based procedure; it simply requires a resolution of $M = 1$ bit for each harmonic phase.

Table II shows the performance of the techniques on example 2 with the restriction to 0 and $\pi$ phase values. The GA finds the lowest-peak amplitude with reductions similar to those shown in example 1. The GA's peak-to-peak factor also improves on a peak factor of 1.21 found by Schroeder using number theory for the special case of white spectra (Schroeder, 1984). Figure 2(a) shows the waveform produced by the GA's solution.

Few applications of flat spectra actually require an even signal. Table III shows how the various techniques perform with unrestricted phase values. Van den Bos's technique and especially the genetic algorithm find very low peak amplitudes, just as they did on example 1. Schroeder's method shows only a slight improvement over its result for the $0/\pi$

TABLE III. Peak amplitudes and peak-to-peak factors for example 2 with unrestricted phases.

| | Peak amplitude | Peak-to-peak factor |
|---|---|---|
| worst | 5.57 | 3.44 |
| $avg_{100}$ random | 2.06 | 1.88 |
| $min_{100}$ random | 1.56 | 1.52 |
| Schroeder | 1.34 | 1.38 |
| van den Bos | 1.11 | 1.07 |
| GA | 1.04 | 1.04 |

TABLE IV. Peak amplitudes and peak-to-peak factors for example 3.

| | Peak amplitude | Peak-to-peak factor |
|---|---|---|
| worst | 2.45 | 2.07 |
| $avg_{100}$ random | 1.96 | 1.80 |
| $min_{100}$ random | 1.60 | 1.57 |
| Schroeder | 1.64 | 1.60 |
| van den Bos | 1.59 | 1.47 |
| GA | 1.42 | 1.42 |

case. Figure 2(b) shows the waveform produced with the GA-derived minimum peak phases.

Since flat spectra play an important role in many applications, we also compare the performance of three of the methods on the phase alignment of flat spectra with 2–40 harmonics. Figure 2(c) shows the results.

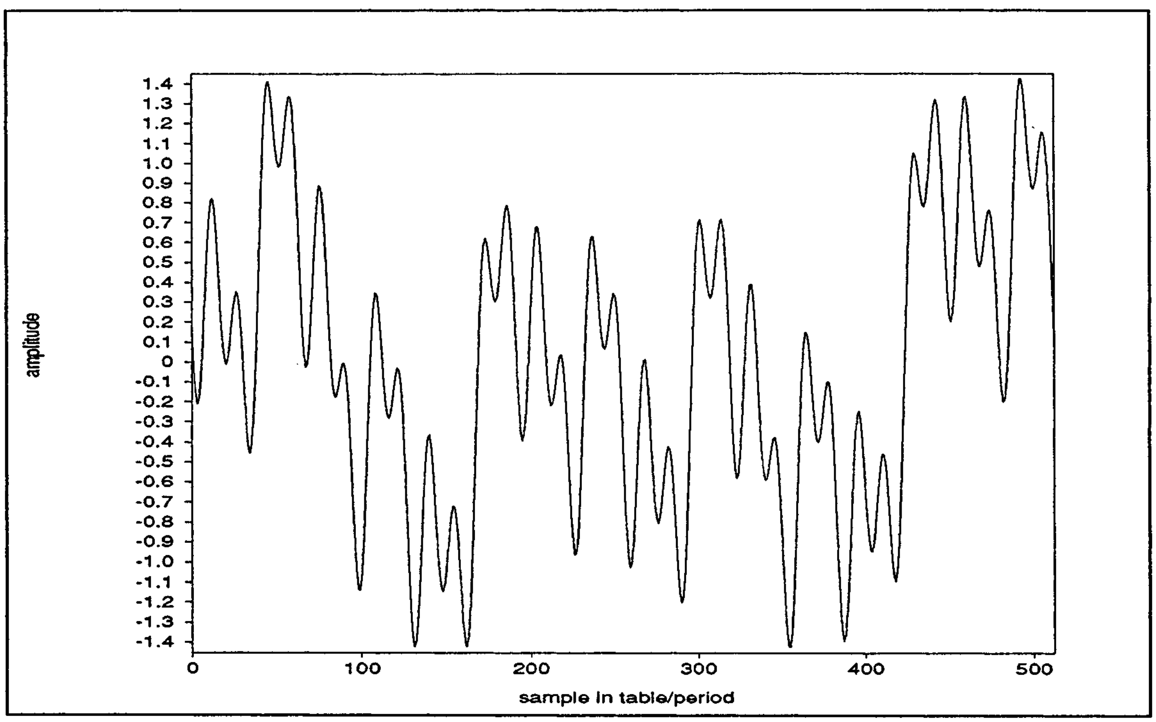
Schroeder's technique consistently returns a minimum peak amplitude of about 1.34 across the range, while van den Bos's method and the GA find much lower values. The general trends of both of the latter two methods move sharply downward for the first seven harmonics, have a relatively flat region for harmonics 8–20, and trace a gradually increasing curve above harmonic 20. The GA-derived peak amplitudes are consistently less than those derived by the van den Bos method and on average are less by 6%.

One might have conjectured from previous results that the peak factor of a waveform could never be less than 1.0, which is the case for the pure sine wave ($K=1$). However, as shown in Fig. 2(c), the GA method found peak amplitudes slightly less than 1.0 for flat spectra with 7, 8, 11, 12, 14, 19, and 20 harmonics, and was very close to 1.0 for several others. The lowest value of peak amplitude found was 0.9795 for the 14-harmonic flat spectrum waveform shown in Fig. 2(d). Obtaining a peak amplitude less than 1.0 requires highly accurate phase values. The real lower bound peak amplitude for normalized spectra remains an interesting question for future study.

**Example 3:** Van den Bos proposed a third example, whose spectral shape does not satisfy Schroeder's assumptions. The spectrum consists of isolated harmonics:

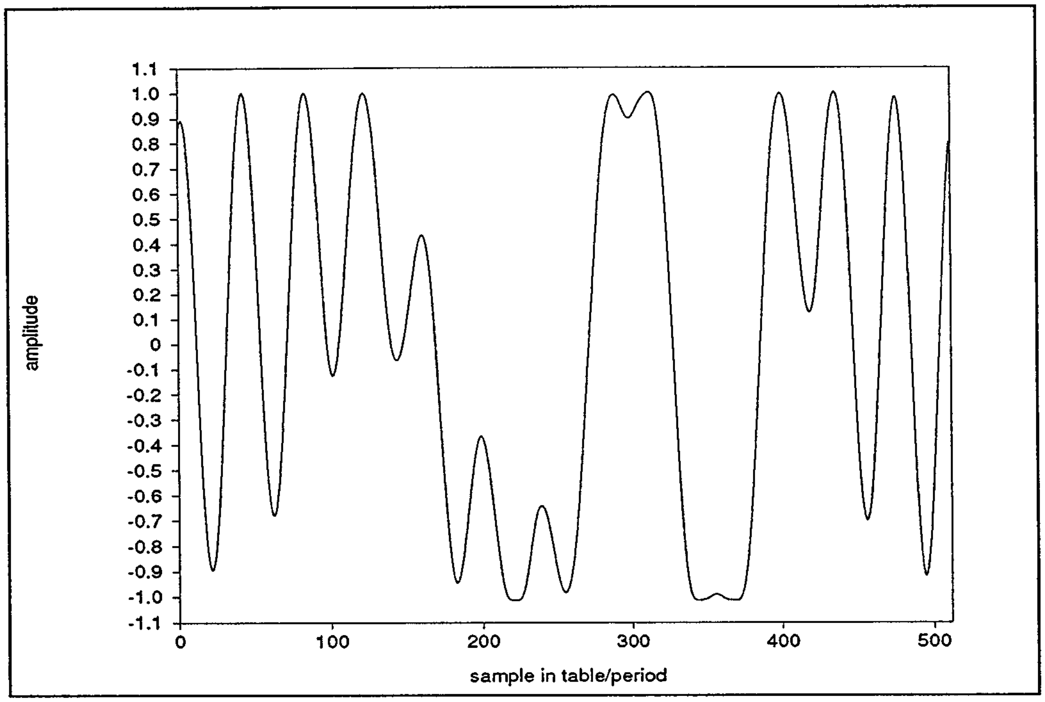
$$a_k = \frac{1}{\sqrt{6}}, \quad k = 2^q, \quad 0 \leqslant q \leqslant 5,$$
$$= 0, \quad \text{otherwise.} \tag{12}$$

Table IV lists the results for the different methods. For this difficult case, the GA shows significant improvement over the methods of Schroeder and van den Bos, especially when




$\{\phi_k\} = \{0.000000, 0.182927, 2.902675, 0.821063, 2.941792, 1.923995\}$ for $k=1,2,4,8,16,32$.

FIG. 3. Waveform and phase values for example 3 with GA-minimized peak amplitude.




$\{\phi_k\} = \{0.000000, 4.021302, 2.055534, 4.571263, 0.210893, 2.523457, 4.763981, 3.105690, 0.823573, 1.659418, 6.004797, 6.031612, 5.786176,$

$5.733224, 0.000000, 0.540874, 2.520059, 0.596777, 6.256370, 3.151593, 0.000000, 2.198932, 2.301884, 4.862836, 1.426486, 2.605495,$

$2.216660, 6.036836, 4.513535, 0.002272, 3.003418, 1.628291, 4.119476, 2.902292, 4.444680, 3.894039, 1.601476, 2.534136, 0.595185,$

$1.918272, 3.239767, 5.487787, 1.521709, 6.168875, 2.347787, 1.255592, 5.325981, 3.331806, 4.751476, 2.202796\}$

FIG. 5. Waveform and phase values for example 4 with GA-minimized peak amplitude.

the peak amplitude criterion is used. Van den Bos's procedure does better with respect to the peak-to-peak factor criterion, but this is because the waveform it finds, for example, 3, has a very asymmetrical shape, tending toward a unipolar signal.

Figure 3 shows the signal produced by the GA-determined phases, using a resolution of $L=512$ samples per phase. Since it does not at all resemble a two-level signal, van den Bos's method would reject it, even though it has a significantly lower peak amplitude than any of the phase sets actually found by his procedure.

**Example 4:** Finally, we test the methods on a spectrum measured from a tenor voice. Figure 4 shows a plot of the spectrum and also lists the normalized partial values, while Table V summarizes the results found by the various techniques. Like example 3, the spectrum here does not fit Schroeder's assumptions, as the results confirm. Schroeder's method does about as well as an average of results obtained using sets of random phases. Both van den Bos's method and the GA technique yield much better performance. Figure 5 shows the waveform generated using the GA-derived phases. The GA-determined peak amplitude of 1.01 is very close to the value for a sine wave.

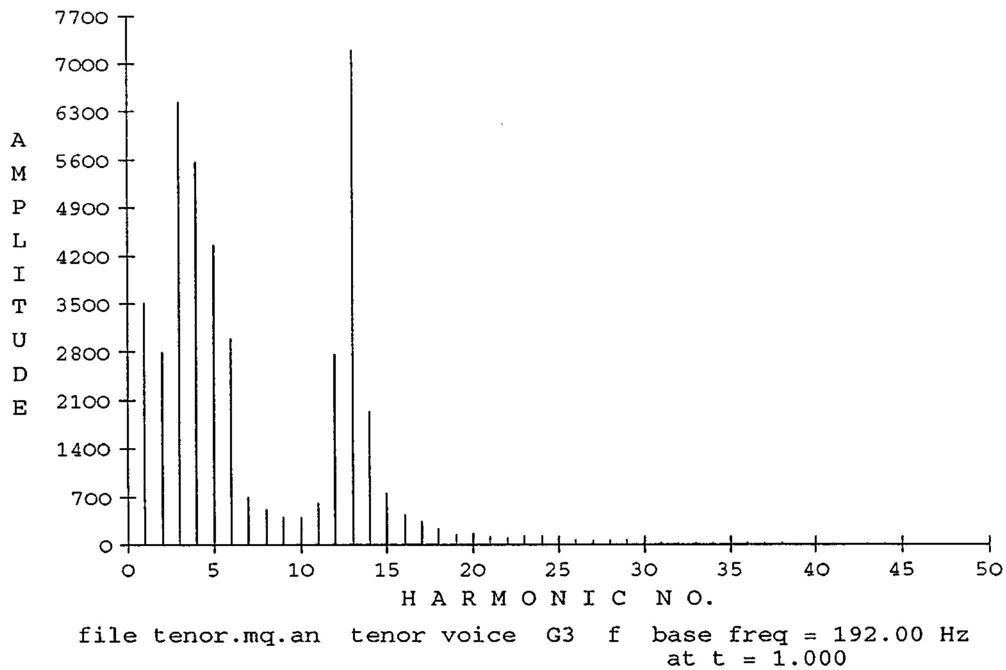
TABLE V. Peak amplitudes and peak-to-peak factors for example 4.

| | Peak amplitude | Peak-to-peak factor |
|---|---|---|
| worst | 3.19 | 1.99 |
| $avg_{100}$ random | 1.80 | 1.65 |
| $min_{100}$ random | 1.34 | 1.31 |
| Schroeder | 1.75 | 1.56 |
| van den Bos | 1.06 | 1.05 |
| GA | 1.01 | 1.01 |

## IV. CONCLUSIONS

The genetic algorithm-based method has proved its effectiveness in designing low peak amplitude signals. The method minimizes the peak amplitudes by systematically exploring many local optima of the phase space in parallel, and has modest software requirements. The method's results are comparatively independent of seed values, and the calculations typically take between a few seconds and a few minutes on a low-cost computer depending on the number of harmonics involved.

The peak amplitudes found with the GA method averaged 6% lower than the best method previously described, the van den Bos method. For some flat spectra we unexpectedly found peak amplitudes less than 1.0, the value for a




$\{a_k\}$ = {0.257775, 0.204637, 0.471793, 0.408101, 0.318536, 0.219296, 0.051306, 0.038333, 0.029904, 0.028365, 0.044709, 0.203024, 0.527277, 0.142117, 0.055117, 0.032176, 0.024920, 0.016638, 0.010334, 0.011654, 0.008649, 0.007696, 0.009162, 0.009308, 0.008282, 0.005570, 0.004324, 0.004471, 0.005717, 0.001906, 0.002272, 0.002272, 0.002565, 0.002565, 0.002419, 0.002126, 0.001906, 0.001759, 0.001612, 0.001612, 0.001612, 0.001539, 0.001539, 0.001173, 0.001099, 0.000660, 0.000440, 0.000293, 0.000147, 0.000073}

FIG. 4. Original tenor spectrum and normalized partial values.

sinewave. This raises some interesting theoretical questions-about the unknown peak factor lower bound for flat spectra. $1/\sqrt{2}$ is probably a lower bound, but can this be proved? What about a tighter bound? And why does the peak amplitude depend on the number of harmonics in such a peculiar way? Further investigation is needed to address these issues.

## APPENDIX

The following phase values were found by the genetic algorithm for the special case of flat spectra for various numbers of harmonics. The number of harmonics precedes its respective phase set. Figure 2(c) gives the peak amplitudes corresponding to these phase sets.

2: {0.000000, 1.570796}

3: {0.000000, 0.281749, 4.135600}

4: {0.000000, 1.288981, 0.975612, 4.904214}

5: {0.000000, 4.226021, 2.090816, 2.904617, 5.018490}

6: {0.000000, 1.912130, 2.265912, 3.734338, 2.033991, 1.016056}

7: {0.000000, 1.006303, 5.826053, 0.568334, 3.727567, 3.116929, 5.768151}

8: {0.000000, 2.650689, 0.641294, 2.877880, 3.780100, 4.705389, 2.848062, 2.155573}

9: {0.000000, 4.242835, 1.216417, 5.760092, 5.746676, 4.521894, 6.252512, 2.045234, 2.921646}

10: {0.000000, 0.429515, 3.160000, 0.137262, 0.470874, 2.566408, 1.168097, 4.706253, 5.680273, 4.289010}

11: {0.000000, 5.664137, 2.855476, 1.969631, 0.392699, 1.309223, 1.961224, 4.194700, 0.771534, 4.994641, 0.118854}

12: {0.000000, 4.950098, 4.133340, 3.340214, 1.928952, 4.817612, 3.350214, 4.900331, 3.993806, 5.772311, 0.971748, 3.579515}

13: {0.000000, 1.045612, 6.110700, 0.543825, 4.745340, 5.933438, 2.596408, 2.821845, 3.916311, 6.232506, 4.293554, 1.856913, 0.364291}

14: {0.000000, 2.837068, 1.034699, 3.818816, 4.368777, 3.906991, 5.475515, 4.804428, 3.646331, 2.075534, 3.783593, 0.098175, 0.554505, 3.853360}

15: {0.000000, 5.299845, 3.583379, 0.062951, 3.460661, 3.133864, 6.037748, 1.951224, 4.049709, 1.932816, 0.299068, 1.056971, 2.963651, 2.876156, 2.890020}

16: {0.000000, 2.207304, 1.093966, 5.041799, 0.560635, 2.245394, 2.713685, 4.032562, 2.704189, 1.912850, 0.009113, 1.517964, 6.060996, 2.110901, 2.545755, 6.274881}

17: {0.000000, 1.627612, 6.185690, 3.843360, 3.101826, 3.112389, 2.659923, 2.830136, 0.733243, 3.770525, 3.081030, 5.600739, 1.085379, 2.752758, 3.622467, 1.874525, 5.310641}

18: {0.000000, 5.573486, 5.988721, 3.403188, 4.024579, 1.861846, 1.846701, 2.893906, 5.218624, 2.549235, 3.605089, 2.340687, 4.665640, 4.097416, 1.135843, 5.709551, 2.149024, 3.154734}

19: {0.000000, 2.346194, 1.472622, 2.071670, 1.623068, 3.828136, 2.377554, 3.005923, 1.033107, 1.189010, 5.757088, 1.343767, 3.348622, 4.693981, 5.080544, 2.143709, 1.767146, 4.967826, 2.738214}

20: {0.000000, 3.867903, 5.009185, 4.652622, 3.296583, 3.861768, 4.861923, 2.139165, 5.438020, 4.503768, 0.251573, 4.413321, 4.849651, 2.425961, 0.349748, 0.473146, 2.968195, 3.718369, 3.286583, 5.599826}

21: {0.000000, 5.164175, 2.587767, 2.251884, 4.761476, 1.933495, 2.915243, 5.141904, 1.245592, 6.100700, 5.749360, 5.454836, 0.889709, 1.380583, 6.117515, 3.933127, 6.077748, 5.841399, 0.969476, 3.772001, 1.751010}

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28: {0.000000, 3.585651, 3.133864, 1.315359, 4.720117, 0.241573, 2.292563, 1.300816, 6.100467, 4.022894, 4.221515, 0.529961, 3.441573, 0.836757, 1.043107, 4.835107, 5.164855, 5.570739, 2.298020, 4.311282, 4.636486, 4.963962, 3.176816, 4.730797, 4.295146, 0.542233, 5.791632, 0.548369}

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